FINA 3103: Intermediate Investment
Class Note: Fixed-Income Securities

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What is Fixed-Income Securities?

Fixed-Income Securities

Fixed-income securities are financial claims of promised cash flows of fixed amount paid at fixed dates

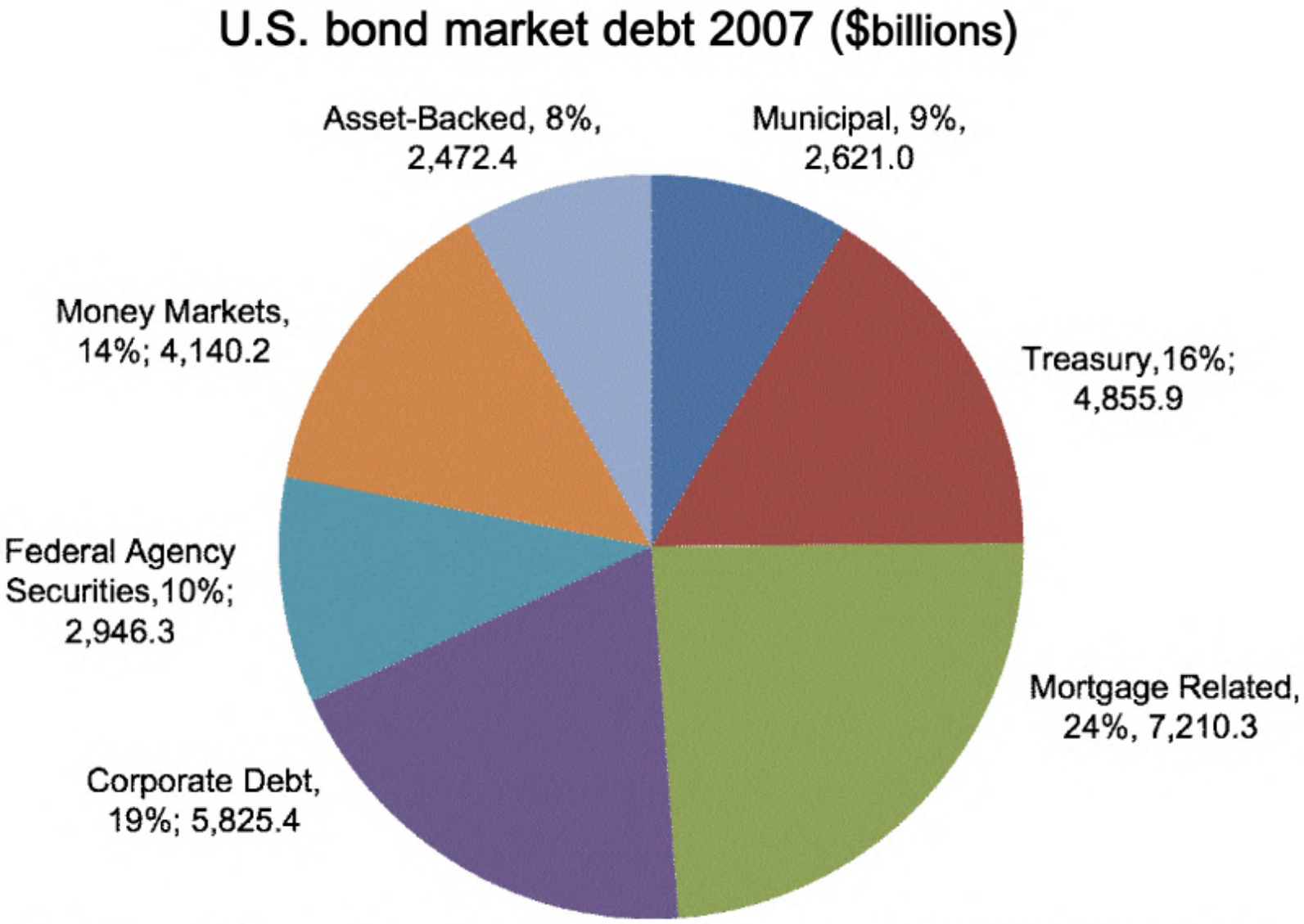
- Leading example: government bonds
  - Pre-determined **coupon** at certain periods
  - **Redemption of face value** in the final period
- Different from stocks?
- Valuation method?

US Government Bond

Classification of Fixed-Income Securities

1. Treasury Securities:
   - US Treasury securities (bills, notes, bonds)
     - Outstanding: $35 trillion
     - Gross issuance $20-25 trillion/y
   - Bunds issued by other countries: UK Gilts, CGB, JGB, etc.
2. Federal Agency Securities ($\sim$ outstanding $9 trillion):
   - Securities issued by federal agencies
   - Mostly mortgage (FHLB, FNMA ...)
3. Corporate Securities ($\sim$ outstanding $11 trillion):
   - Commercial papers (CP)
   - Medium-term notes (MTN)
   - Corporate bonds
4. Municipal Bonds (Munies)
5. Mortgage-Backed Securities (non-agency, pre-GFC)
6. Asset Backed Securities (non-mortgage), ...

U.S. Bond Markets (Secondary)



U.S. Bond Markets (Secondary)

Outstanding U.S. bond market debt ($ billions)

°°	Municipal	Treasury	Mortgage Related	Corporate Debt	Federal Agency Securities	Money Markets	Asset-Backed	Total
1998	1,402.7	3,542.8	2,955.2	2,708.5	1,300.6	1,977.8	731.5	14,619.1
1999	1,457.1	3,529.5	3,334.2	3,046.5	1,620.0	2,338.8	900.8	16,226.9
2000	1,480.5	3,210.0	3,565.8	3,358.4	1,854.6	2,662.6	1,071.8	17,203.7
2001	1,603.6	3,196.6	4,127.6	3,836.4	2,149.6	2,587.2	1,281.1	18,782.1
2002	1,763.0	3,469.2	4,686.4	4,099.5	2,292.8	2,545.7	1,543.3	20,399.9
2003	1,900.7	3,822.1	5,238.6	4,458.4	2,636.7	2,519.9	1,693.7	22,270.1
2004	2,030.9	4,257.2	5,455.8	4,785.1	2,745.1	2,904.2	1,827.8	24,006.1
2005	2,226.0	4,517.3	5,915.6	4,960.0	2,613.8	3,433.7	1,955.2	25,621.6
2006	2,403.2	4,689.8	6,492.4	5,365.0	2,660.1	4,008.8	2,130.4	27,749.7
2007	2,621.0	4,855.9	7,210.3	5,825.4	2,946.3	4,140.2	2,472.4	30,071.5
2008Q1	2,657.0	4,995.8	7,397.0	5,905.6	2,984.2	4,125.9	2,480.3	30,545.8

Sources: U.S. Department of Treasury, Federal Reserve System, Federal Agencies, Thomson Financial, Bloomberg, SIFMA

Bond Markets Players and Their Roles

1. Government:
   - 1.1 Dpt. of Treasury (Ministry of Finance) issues new bonds (why?)
   - 1.2 Central bank (FR, BOE, BOJ, etc) trades bonds (why?)
2. Banks, Insurance, Pension funds (why?)
3. Securities companies, brokerage firms (why?)
4. What about bonds for retail holding (you and me)?

Fixed-Income Securities

- Specified by
  1. **Bond value**: market price of the bond (or return rate)
  2. **Principal (face value)**: you get this value on maturity
  3. **Maturity**: timing of redemption
  4. **Coupon**: you receive this value periodically

Fixed-Income Securities

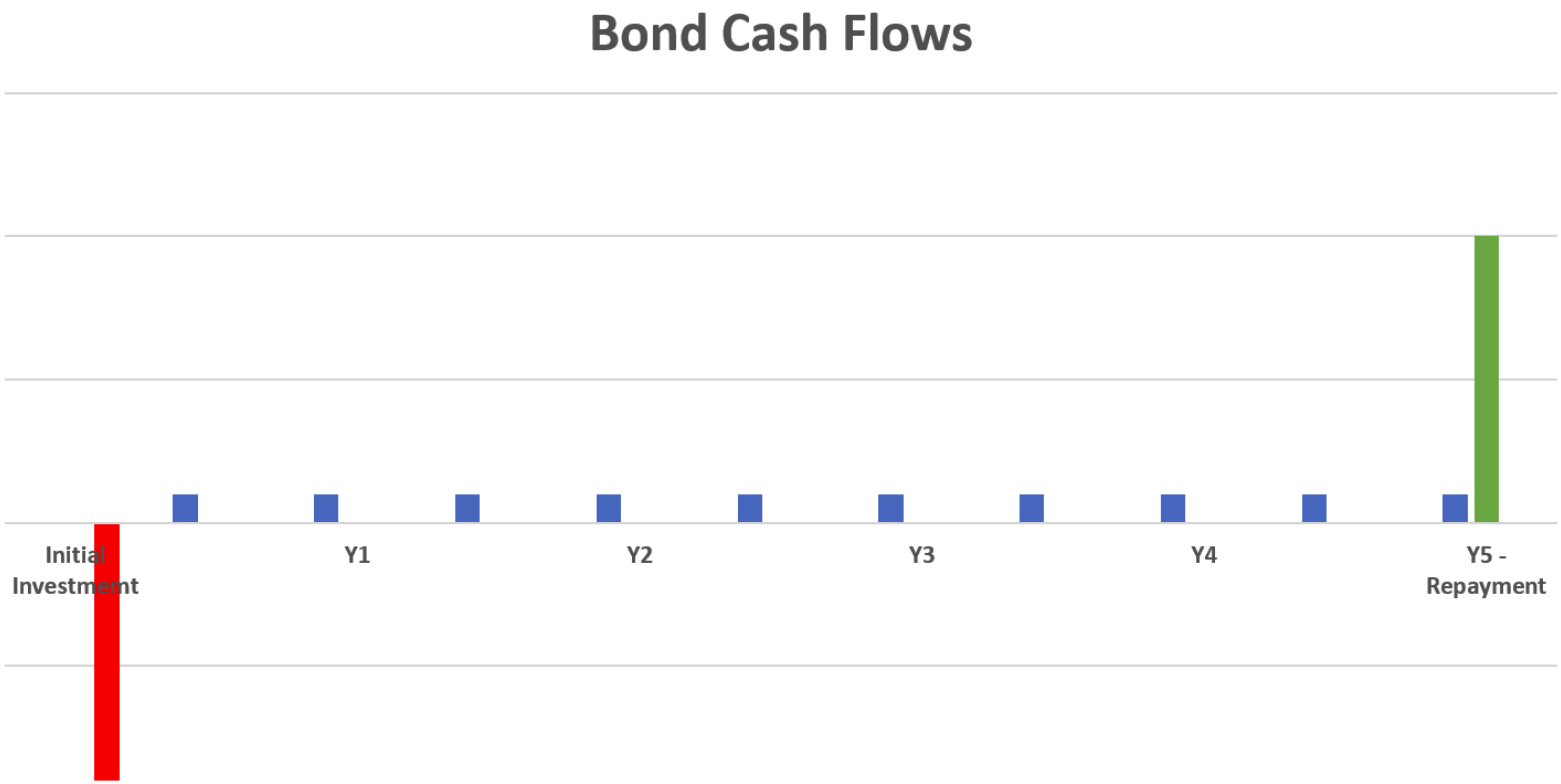
Cash Flow of Fixed-Income Security

Consider a 3-year bond with FV \$1,000 and annual coupon rate 5%. What does your cash flow look like?

- You pay P today to buy this bond
- One year later, you receive 5% of FV = \$50
- In the end of 2nd and 3rd year, you receive \$50 each
- In the end of 3rd year, you also receive FV = \$1,000
- **How much you pay (P)?** —Need a valuation scheme

Fixed-Income Securities

- Cash-flow structure is different from stocks



Valuation of Fixed-Income Security

Q. Assuming no risk, what is the **current value** of future CF?

- Rich set of fixed-income securities traded in the market, providing information to evaluate future CF
- In the market, valuation information is given in many different forms:
  1. **Yield to maturity**
  2. **Spot interest rates**
  3. **Prices of discount (zero-coupon) and coupon bonds**
  4. **Forward interest rates**

Bond Valuation

Bond is priced as the discounted CF

- C = coupon, FV = face value you receive at maturity T
- Assuming constant interest rate r:

$$P = \underbrace{\sum_{t=1}^{T} \frac{C}{(1+r)^t}}_{\text{PV of coupon payment}} + \underbrace{\frac{FV}{(1+r)^T}}_{\text{PV of redemption}} \tag{1}$$

- Theoretical equation (actual P may deviate)
- Note: (C, FV, T, P, r) are over-identified
  - Observing four of them, last one is determined

Zero-Coupon Bond

Zero-Coupon Bond

Bond with $C = 0$ is called zero-coupon (or discount) bonds

- No coupons, single payment of principal at maturity
- Trades at a "discount" to face value

$$P_{0,T} = \underbrace{\frac{FV}{(1+r)^T}}_{\text{PV of redemption}} \tag{2}$$

  - Also known as STRIPS (Separate Trading of Registered Interest and Principal of Securities)

Valuation of Coupon Bond by Discount Bonds

Coupon Bond Valuation

Idea: Coupon payment $C = \$1$ in each period can be seen as the redemption of discount bond with $FV = \$1$

- $P_{0,t}$ = price of ZCB with FV = \$1 and maturity at t
- The price of coupon bond in (1) is

$$P = \sum_{t=1}^{T} C \times \frac{1}{(1+r)^t} + FV \times \frac{1}{(1+r)^T}$$

$$= \sum_{t=1}^{T} C \times P_{0,t} + FV \times P_{0,T}$$

  - Applied (2) with $FV = 1$
- Ex: Bond with $(T, C, FV) = (3\text{y}, 5\%, \$100)$ is replicated by ZCBs

Yield to Maturiy

Yield to Maturity

- Suppose that we observe (T, C, FV, P) but r is unknown.

YTM

Yield to maturity (YTM) is a constant interest rate y that makes DCF of bond equal to its price P.

- YTM is the rate $r = y$ that solves

$$P = \sum_{t=1}^{T} \frac{C}{(1+y)^t} + \frac{FV}{(1+y)^T}$$

- (C, T, FV) are fixed. Use market P and solve for y
- Note: typically no closed-form solution for y
  - Numerical methods, e.g., Excel function $y =$`RATE(T,FV*C%,-P,FV)`

Yield to Maturity: Example

Example

A bond with maturity $T = 20$, face value $FV = \$1{,}000$, and annual coupon payment at $C = \$35$ (i.e., 3.5%) is traded at $P = \$950$. What is the yield to maturity?

- Need to solve (numerically)

$$\$950 = \sum_{t=1}^{20} \frac{\$35}{(1+y)^t} + \frac{\$1{,}000}{(1+y)^T}$$

$$\Rightarrow y \approx 3.86\%$$

- Or use computation: `RATE(20,1,000*0.035,-950,1,000)`

Yield to Maturity

- y can be seen as (compounded) return rate of bond holding
- E.g., you bought a bond with $(T, FV, C) = (2, \$100, 5\%)$ at P
- What is return rate (or show that the rate is y)?
  - Caveat: y may not be realized return

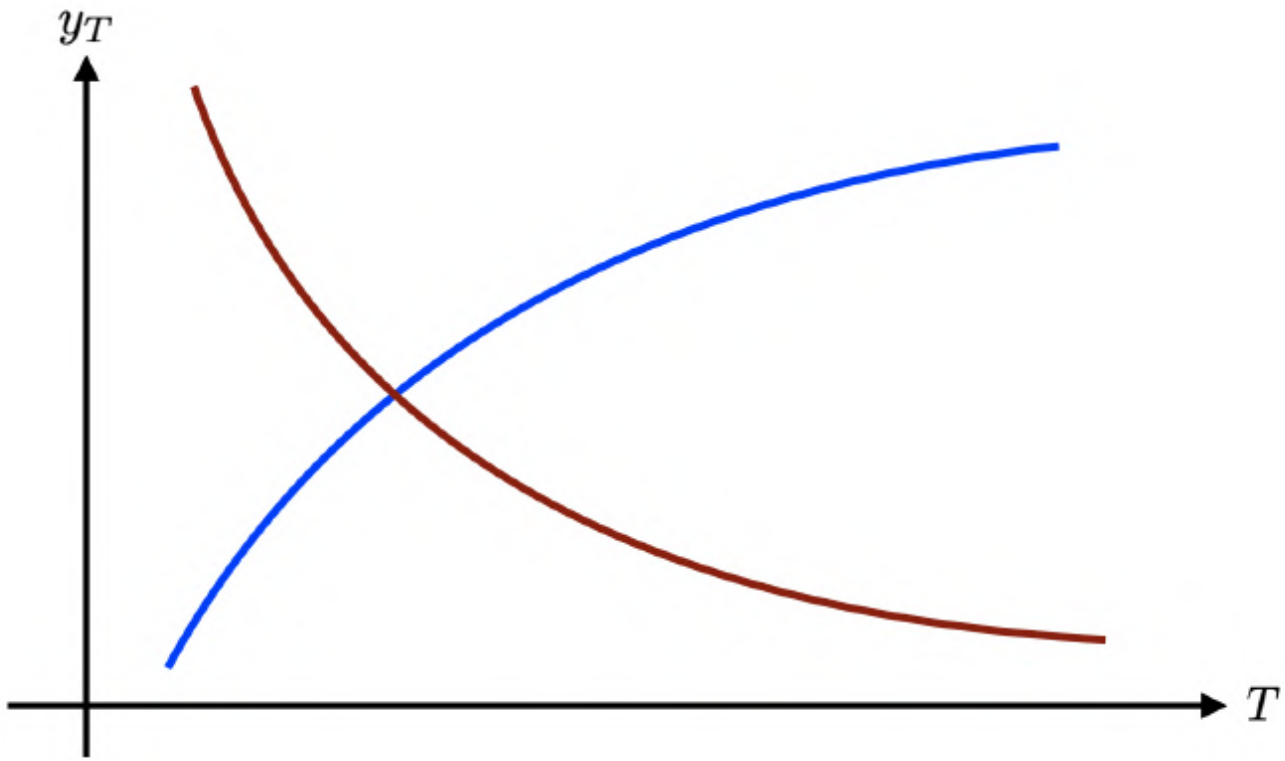
Yield Curve

$$P = \sum_{t=1}^{T} \frac{C}{(1+y)^t} + \frac{FV}{(1+y)^T}$$

- Yield y is specific to each bond
- **Yield curve**: plot of y against T focusing on sovereign bonds
  - US bonds with different T (same FV, C) are traded at different P
  - Compute y for $T = 1$ and $T = 10$ US bonds and plot against T
- What does it look like?

Yield Curve: Upward- or Downward-Sloping?

- Which one makes more sense, especially macro risk?



Yield Curve: Upward- or Downward-Sloping?

Example

Today (Jan 1st, 2026), two investments with same FV are available:

1. Two-year ZCB at $P_{0,2}$ and return $y(2)$
2. One-year ZCB at $P_{0,1}$ with $y(1) = 1\%$; and roll it over on Jan 1st, 2027 at $P_{1,1}$ with $y'(1)$.

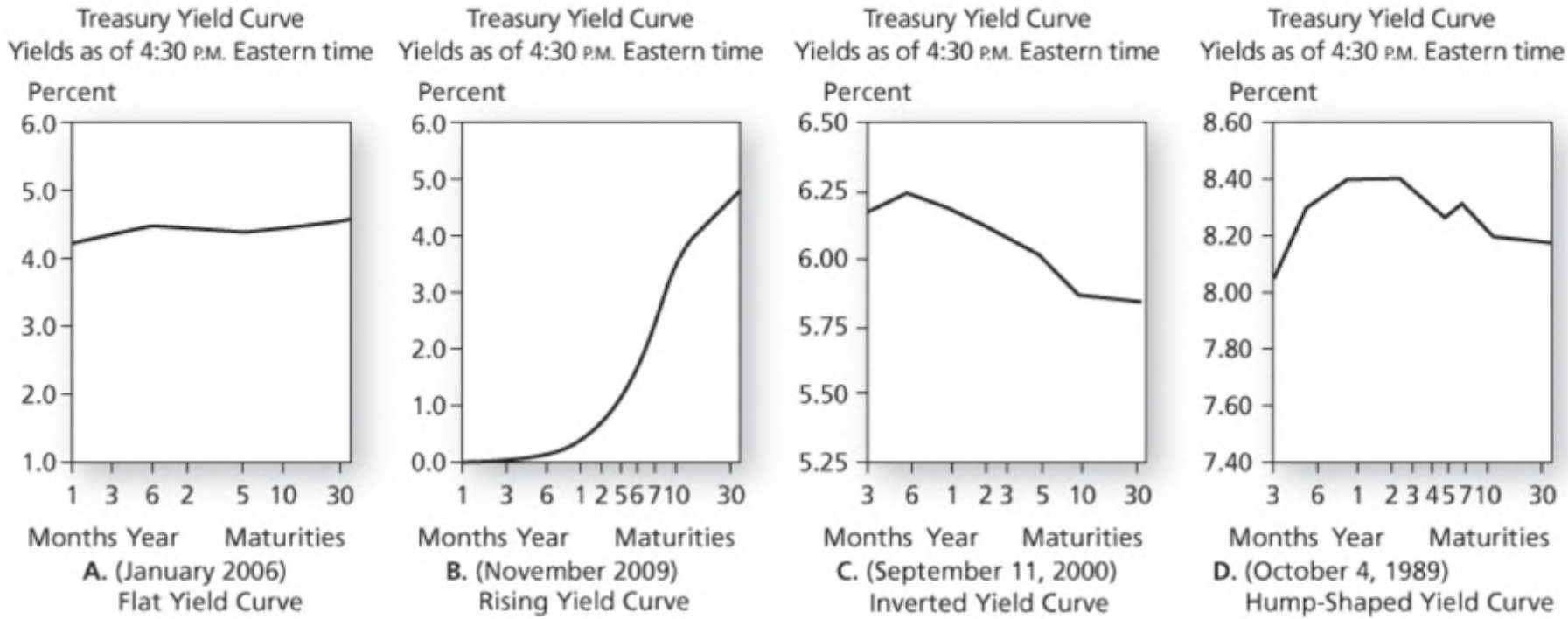
The government has announced to increase the short-term rate for 2027, so $y'(1) = 5\%$. Assuming no arbitrage, is the yield curve upward- or downward-sloping?

- Assuming no risk, no-arb condition is

$$(1 + y(2))^2 = (1 + y(1))(1 + y'(1))$$

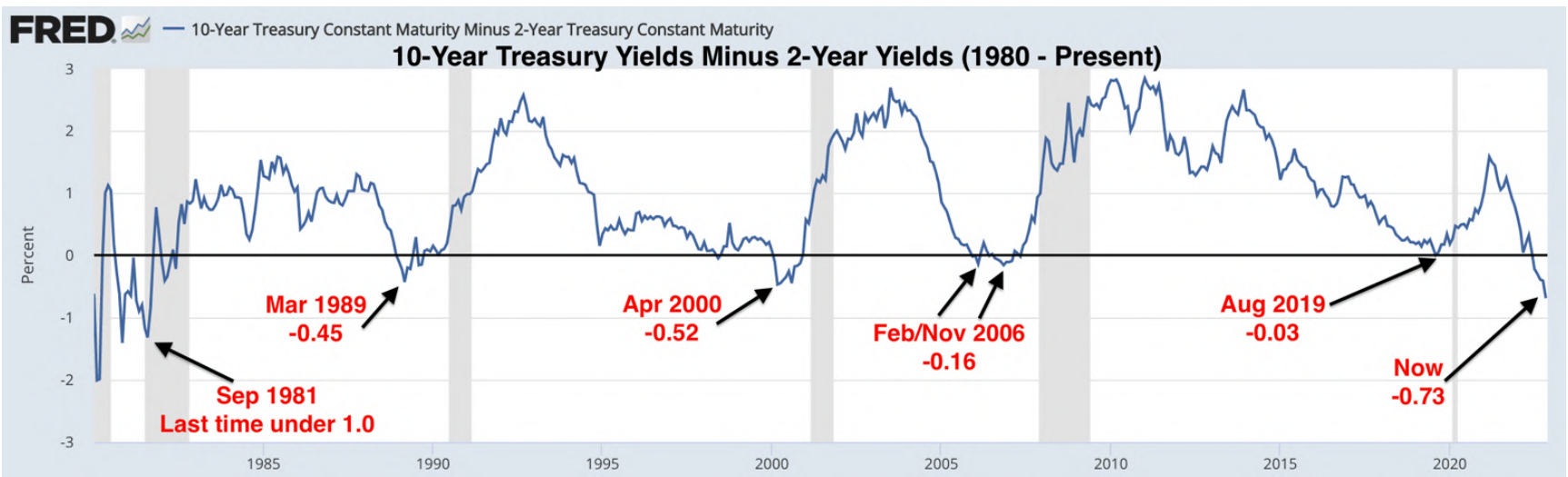
- $y(2) > y(1) \Leftrightarrow y'(1) > y(1)$

The Actual Yield Curve



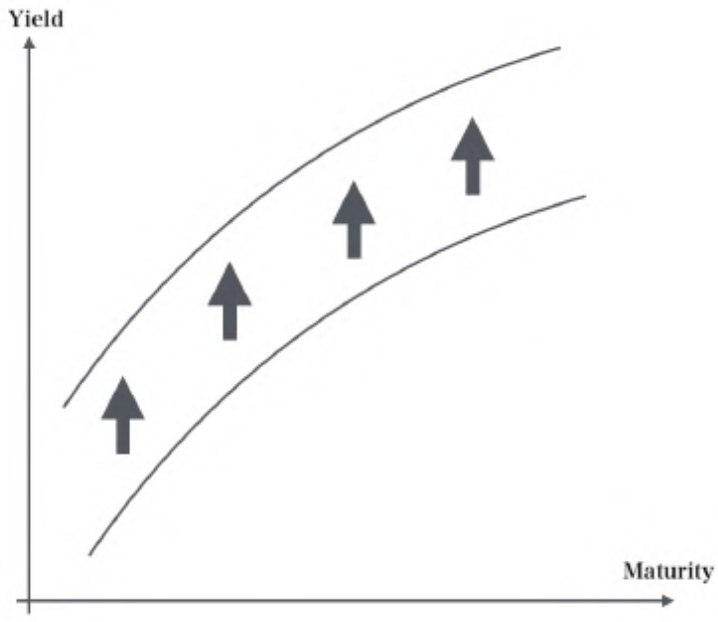
Inverted Yield Curve

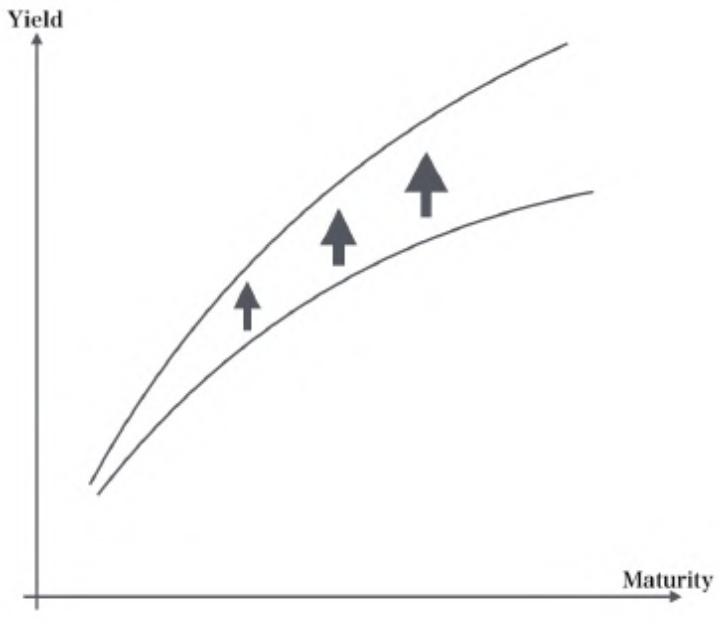
- Shape of the yield curve often indicates investor expectation
- Flat and inverted curves?
  - Investors demand the same or lower return for long-term bonds
  - Investors expect declines in short-term rates in the future, typically associated with recessions



Changes in Yield Curve

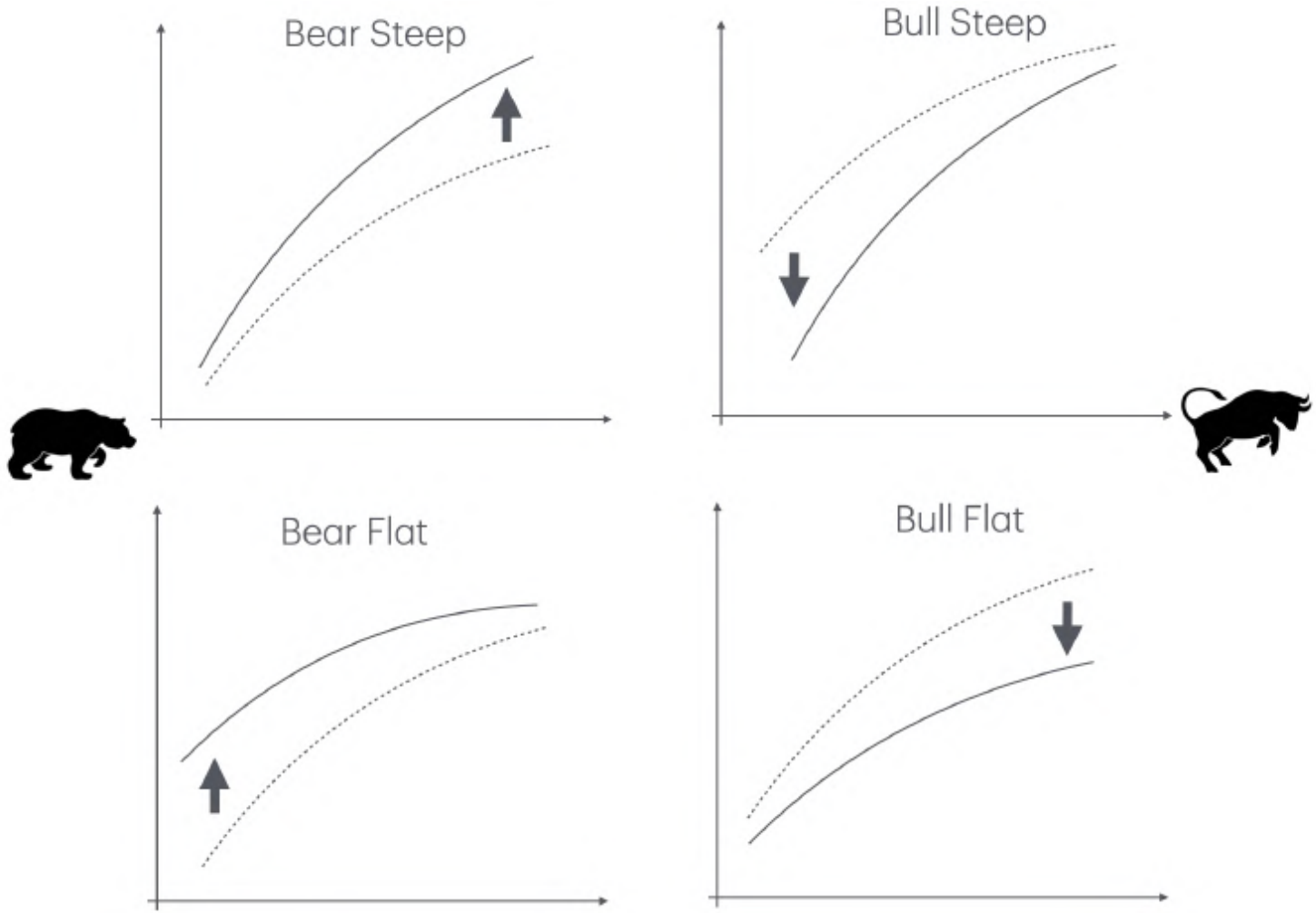
- A news article says "the interest rate is going up/down."
- y and P move in opposite directions
  - Bullish market = $P \uparrow$ and $y \downarrow$
- Several possible patterns:





Changes in Yield Curve

- Investors often use the terms “steep” and “flat” to describe the change



Spot Rate

T-Year Spot Rate

What if we allow different r for each period?

- Spot rate r_t: interest rate applied to period $(t-1, t)$
  - Universal rates: applicable to all bonds
  - Computed based on ZCB
- ZCB should be priced by

$$P = \frac{FV}{(1+r_1)(1+r_2)\cdots(1+r_T)}$$

- But can we observe future r_t?

T-Year Spot Rate

- Instead, consider $r_{0,T}$ such that

$$\underbrace{\frac{FV}{(1+r_{0,T})^T}}_{\text{we define } r_{0,T}} = P = \frac{FV}{(1+r_1)(1+r_2)\cdots(1+r_T)}$$

$$\therefore r_{0,T} = \left(\frac{FV}{P}\right)^{1/T} - 1 = \left[(1+r_1)(1+r_2)\cdots(1+r_T)\right]^{1/T} - 1$$

- $r_{0,T}$ is called T-year spot rate
  - red: what you cannot observe perfectly
  - blue: observable (P, FV, T)
- Observable rates that capture the average of unobservable $\{r_t\}_{t=1}^T$

Example: T-Year Spot Rate

Example

ZCBs with $T = \{1,2,5,10\}$ and $FV = \$1$ are traded at the following prices. What is the T-year spot rate?

Maturity	1	2	5	10
Price	\$0.967	\$0.927	\$0.797	\$0.605

- $P_{0,T} = 1/(1 + r_{0,T})^T$
- E.g., 5-year Spot Rate: $0.797 = 1/(1 + r_{0,5})^5 \Rightarrow r_{0,5} = 4.64\%$

Yield to Maturity vs. Spot Rates

Example

Two bonds are available:

Bond	Maturity	FV	C	y
A	2	\$100	8%	11.9%
B	3	\$100	10%	13.7%

The spot rates are $(r_{0,1}, r_{0,2}, r_{0,3}) = (10\%, 12\%, 14\%)$. What are the prices of these bonds?

- Prices based on YTM y:

$$P_A = \frac{\$8}{1.119} + \frac{\$108}{(1.119)^2} = \$93.4$$

$$P_B = \frac{\$10}{1.137} + \frac{\$10}{(1.137)^2} + \frac{\$110}{(1.137)^3} = \$91.3$$

Yield to Maturity vs. Spot Rates (cont'd)

Example

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$$P_B = \frac{\$10}{1.10} + \frac{\$10}{(1.12)^2} + \frac{\$110}{(1.14)^3} = \$91.3$$

Term Structure and Forward Rate

Term Structure

- The t-year spot rate summarizes information about time-series of **future** short rates

$$\underbrace{1 + r_{0,t}}_{\text{compute from ZCB}} = \underbrace{[(1 + r_1)(1 + r_2) \cdots (1 + r_t)]}_{\text{future short rates}}^{1/t}$$

- The relationship is called term structure
- 1-year rates applied in future periods are called forward rates: $r_t = f_t$
  - Assuming uncertainty, $\{r_t\}_{t=1}^{T}$ is no longer fixed
  - f_t involves future expectation by investors

Forward Rate

Forward trade

Consider a forward transaction to borrow money in the future:

1. Terms of the transaction are agreed on today, $t = 0$
2. Loan will be made at some future timing, $t = t_1$
3. Repayment at $t = t_2 (> t_1)$

- If you are lending, you incorporate future expectation

Estimating Forward Rate

- Back out f_t using observable prices:

$$P_{0,t-1} = \frac{FV}{(1+f_1)\cdots(1+f_{t-1})}$$

$$P_{0,t} = \frac{FV}{(1+f_1)\cdots(1+f_t)}$$

$$\Rightarrow \frac{P_{0,t-1}}{P_{0,t}} = 1 + f_t$$

- Interest rate between $(t-1, t)$ from today's perspective
- Market-based $P_{0,t} \Rightarrow f_t$ reflects investors' expectation
  - f_t could be different from actual realization of r_t

Forward Rate: Example

Forward Contract

Suppose that discount bond prices and spot rates are as follows.

T	1	2	3	4
P_T	0.9524	0.8900	0.8278	0.7629
$r_{0,T}$	0.05	0.06	0.065	0.07

A customer asks for a forward contract to borrow \$20M three years from now for one year. Can you (as a bank) quote a rate for this forward loan?

- Compute f_4 and quote it

Example (cont'd)

Forward Contract

Suppose that you are a borrower and there is no bank available. Instead, you go to bond market to finance your expenditure.

- Buy $FV = \$20\text{M}$ of 3-year discount bonds. The cost is

$$\text{Cost} = \$20\text{M} \times P_3 = \$16{,}556{,}000$$

- Finance it by short-selling 4-year discount bonds of FV

$$FV = \text{Cost}/P_{0,4} = \$21{,}701{,}403$$

Example (cont'd)

- This position generates the following cash flow (in $million)

Position	$t = 0$	1-2	3	4
Long (3-y bonds)	-16.556	0	20.000	0
Short (4-y bonds)	16.556	0	0	-21.701
Total CF	0		20.000	-21.701

- The yield (rate of return) for this synthetic bond is

$$r = \frac{21.701}{20.000} - 1 = 8.51\%$$

Next Class

- We have derived three types of rates:
  1. Yield to maturity, y
  2. T-year spot rate, $r_{0,t}$
  3. Forward rate, f_t
- We have assumed no uncertainty so far
  - Default risk, duration, and convexity