FINA 3103: Intermediate Investment
Class Note 4: Risk, Return, and Capital Allocation (cont'd)

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Our objective

To derive the optimal investment strategy, we need

1. Criteria to evaluate your happiness from return of investments
   $\Uparrow$ today!
2. Understanding of risk-return performance of *individual* stocks
   $\Uparrow$ class 3
3. ... of *combination* of assets (portfolio)
   $\Uparrow$ class 5

Money can really buy you happiness?

Money can really buy you happiness?

- Jack Ma on CNBC article: "I am not happy about being the richest man in China."
- Why? How can we measure happiness?
  - Especially investment-related happiness?

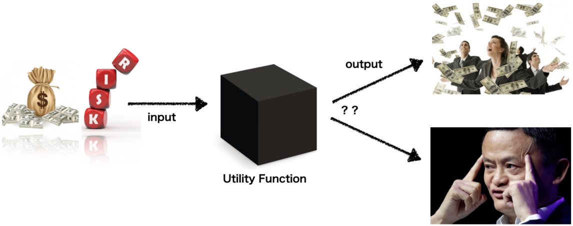
Mean-Variance Investors

Assumptions on investor preference

We want to describe mean-variance investors who

- like higher mean return (or expected return), μ
- but don't like SD (or variance) of return, σ and σ^2
- and care only about risk and return (no higher-order moments)

Need a function $(\mu, \sigma) \Rightarrow$ happines



Prep for Derivation

Assumptions on investor preference

- Assume that an investment generates return r with

$$\mathbb{E}[r] = \mu, Var(r) = \sigma^2$$

- If you invest \$1, what is your end-of-period wealth, \$$V$?

$$V = \$1 \times (1 + r)$$

- Expected value and variance of V:

$$\mathbb{E}[V] = \mathbb{E}[1 + r] = 1 + \mu,$$

$$Var(V) = Var(1 + r) = \sigma^2.$$

- Normalize your budget to \$1 (what if \$$W$?)
- Assume that you derive utility from \$$V$

Utility Function

Mean-Variance Utility

Mean-variance utility

The investor with Mean-Variance utility with parameter γ derives following utility from V.

$$U \equiv \mathbb{E}[V] - \frac{\gamma}{2} Var(V) = 1 + \mathbb{E}[r] - \frac{\gamma}{2} Var(r)$$
$$= 1 + \mu - \frac{\gamma}{2} \sigma^2$$

- Your happiness (= utility) is a function of μ and σ^2
  - Higher moments are irrelevant (e.g., kurtosis, skewness)
- If r is normally distributed, derived from CARA utility
- Caveat: it does not fully capture the real world (why?)

Risk Aversion

Mean-Variance Utility

$$U \equiv \mathbb{E}[1+r] - \frac{\gamma}{2}Var(r) = 1 + \mu - \frac{\gamma}{2}\sigma^2$$

- What is γ?
- **Risk aversion** parameter
  - $\gamma > 0$: risk averse (usual case)
  - $\gamma = 0$: risk neutral
  - $\gamma < 0$: risk loving
- If $\gamma > 0$, utility decreases with risk

MV Utility Function: Example

Example

Consider asset A with 10% expected return and variance of 0.02.
(1) What utility level does an investor with risk-aversion parameter 4 derive?
(2)How would you compare this investor to less risk averse investor, say $\gamma = 2$?

- The first investor derives

$$U =$$

- The second investor derives $U =$
- Given (μ, σ), risk tolerant investors derive ______ utility

Certainty Equivalent (CE)

Certainty Equivalent Return

Hypothetical risk-free return that makes you indifferent between investing in the risk-free asset and risky asset

- Asset 1: risky asset with exp. return $\mu = \mathbb{E}[r]$ and risk $\sigma^2 = Var(r)$
  - Utility is $U_1 = 1 + \mu - \frac{\gamma}{2}\sigma^2$
- Asset 2: risk-free asset with return r_f
  - Utility is $U_2 = 1 + r_f$
- **Certainty Equivalent rate** is r_f (denote it as r_{CE}) that makes you indifferent:

$$r_{CE} = \mu - \frac{\gamma}{2}\sigma^2.$$

Certainty Equivalent: Example

Example

There are two assets:
Asset A: expected return $\mu_A = 30\%$ and variance $\sigma_A^2 = 0.15$.
Asset B: expected return $\mu_B = 20\%$ and variance $\sigma_B^2 = 0.05$.
Suppose that an investor has MV utility and is indifferent between these two assets.
(i) What is the investor's risk aversion, γ?
(ii) What is the investor's certainty equivalent return?

Different Investors, Different Choices

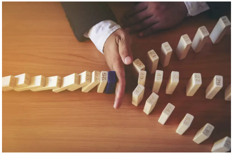
Example

Asset A has expected return of $\mu_A = 22\%$ with variance $\sigma_A^2 = 0.09$. A risk free asset with return $r_f = 4\%$ is also available. Assume that you can invest in only one asset type. Describe an investor's investment choice by using her risk-aversion parameter γ.

- Investing in asset A gives: $U_A = 1.22 - \frac{\gamma}{2} \times 0.09$
- Risk free asset: $U_f = 1.04$

Hedging, Speculation, and Gambling

- **Hedging**: a risk management strategy to offset future losses
- By taking an position in a related asset or buying insurance at some premium
  - Ex., homeowner's insurance to hedge against fire
  - Ex., future contract to buy rice to avoid price fluctuations
- You have **high risk aversion**



Hedging, Speculation, and Gambling

- **Speculation**: risk averse but tolerant
  - Accept risky position to boost your return
  - Ex., buy on margin (taking leverage)
  - Ex., hold cryptocurrency or other highly volatile assets
  - You have **low risk aversion**
- **Gambling**: taking risk gives you higher utility
  - Increase your risk without demanding an increase in expected return
  - You are **risk loving**

Visualizing Utility Function

(Toward Optimal Investment)

Utility Indifference Curve

Utility Indifference Curve: the risk-return combinations that generate the same utility level (given γ)

Indifferent assets

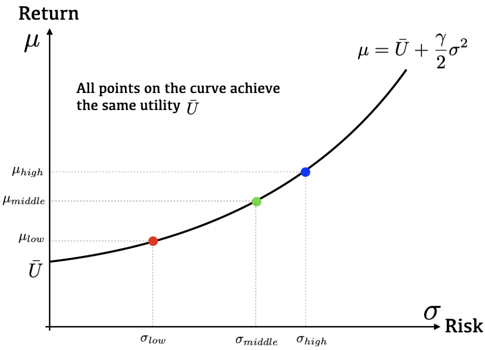
Suppose that $\gamma = 2$.
(i) What level of utility does an asset with $(\mu, \sigma^2) = (50\%, 30\%)$ generate?
(ii) Provide an example of other assets that yield the same utility.

- $U = 1.5 - \frac{2}{2}0.3 = 1.2$
- For example, $(\mu, \sigma^2) = (40\%, 20\%), (60\%, 40\%), \cdots$ all provide the same utility of $U = 1.2$
- Set of such assets draw a curve on (μ, σ) plane

Utility Indifference Curve

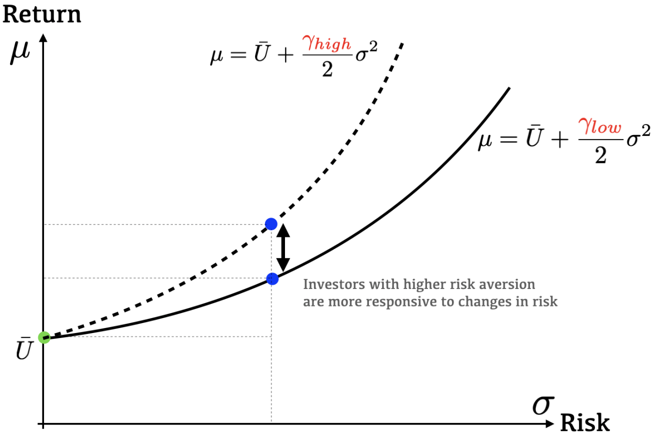
With $\bar{U} \equiv U - 1$ being fixed, plot $\mathbb{E}[r] = \mu$ against $Var(r) = \sigma^2$:

$$\bar{U} = \mu - \frac{\gamma}{2}\sigma^2 \Rightarrow \mu = \bar{U} + \frac{\gamma}{2}\sigma^2$$



- With $\gamma > 0$, risk-return relationship is upward sloping. Intuition?

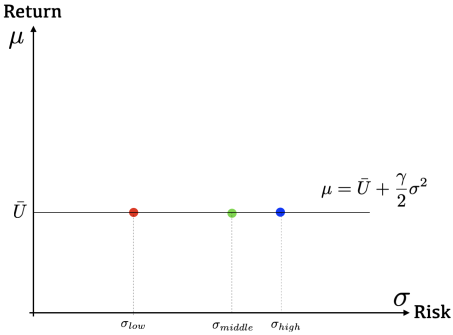
Utility Indifference Curve



- A higher $\gamma > 0$ makes the curve steeper. Intuition?

Utility Indifference Curve

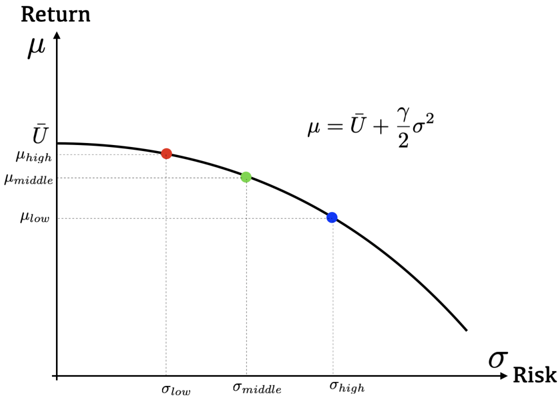
What if an investor is risk neutral?



- Risk neutral investors have $\gamma = 0$ and the curve is $\mu = \bar{U}$
- Not affected by risk, leading to a flat curve (i.e., straight line)
- Do not demand higher (or lower) return even if σ changes

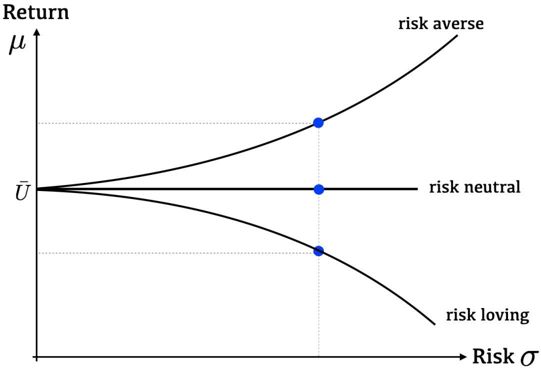
Utility Indifference Curve

What if an investor is risk loving?



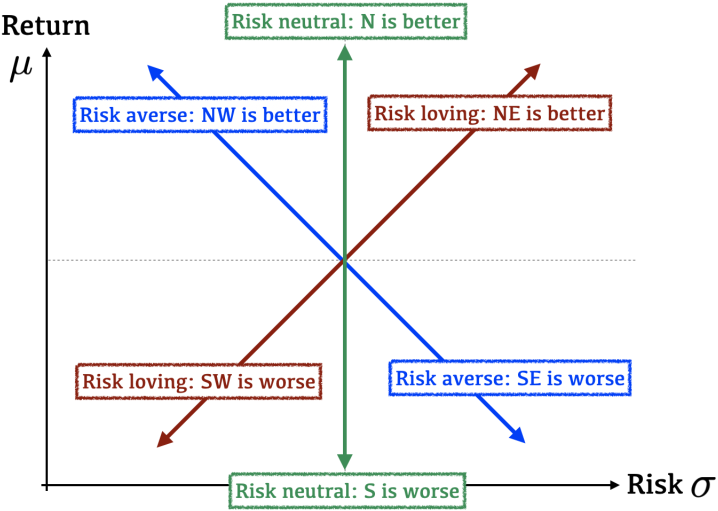
- With $\gamma < 0$, the curve is downward sloping
- The higher the risk, the lower the return the investor demands

Utility Indifference Curve



- Given the risk, different levels of reward you demand depending on risk aversion

Mapping Mean-Variance Criterion



Maximizing Utility Function

Optimal Investment using MV Utility

Question

Risky asset with return r and risk-free asset with r_f are available. Mean and variance of r is (μ, σ^2). You have \$1 and allocate \$$x$ to buy risky asset and allocate $1 - x$ to risk-free asset. If you have MV utility with γ, what is the optimal value of x?

- Called mean-variance optimum

Derivation

1. Write down your end-of-period wealth (V) using your control variable (x)
   - since you save $1 - x$ and invest x...

$$V = \underbrace{x(1+r)}_{\text{risky return}} + \underbrace{(1-x)(1+r_f)}_{\text{safe return}}$$
$$= 1 + r_f + x(r - r_f)$$

2. Write down your utility (U) using x

$$U = \mathbb{E}[V] - \frac{\gamma}{2} Var(V)$$
$$= \mathbb{E}[1 + r_f + x(r - r_f)] - \frac{\gamma}{2} Var(xr) \Leftarrow r_f \text{ is deterministic}$$
$$= 1 + r_f + x(\mu - r_f) - \frac{\gamma}{2} x^2 \sigma^2$$

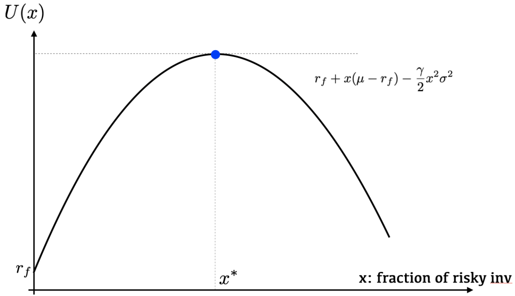
Cont'd

$$U(x) = 1 + r_f + x(\mu - r_f) - \frac{\gamma}{2} x^2 \sigma^2$$

3. What is x that maximizes your utility?

$$\max_x \left[r_f + x(\mu - r_f) - \frac{\gamma}{2} x^2 \sigma^2 \right]$$

So draw a picture of U wrt x



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4. We see that x^* is the point at which the slope of U is flat $(= 0)$
   4.1 Slope of U at x is equivalent to the first-order derivative:
   $$\frac{dU}{dx} = \mu - r_f - \gamma x \sigma^2$$
   4.2 This slope must be zero (= flat) at x^*:
   $$\mu - r_f - \gamma x^* \sigma^2 = 0 \tag{1}$$
   4.3 Solve (1) for x^* to get the optimal investment
   $$x^* = \frac{\mu - r_f}{\gamma \sigma^2}$$

   - Done!

Mean-Variance Optimum

- How much U will you get?

$$U^* \equiv U(x^*) = 1 + r_f + x^*(\mu - r_f) - \frac{\gamma}{2} x^{*2} \sigma^2$$
$$= 1 + r_f + \frac{(\mu - r_f)^2}{2\gamma\sigma^2} \tag{2}$$

  - U^* is called *indirect utility*
- Blue part: additional utility from optimal investment

Mean-Variance Optimum

Optimal Investment

The optimal amount of money you invest in risky asset is

$$x^* = \frac{\mu - r_f}{\gamma \sigma^2}$$

- Buy the risky asset if $\mu > r_f$ (and vice versa)
- Buy more if
  - the excess return increases ($\mu - r_f \uparrow$)
  - the asset is less risky ($\sigma^2 \downarrow$)
  - you are less risk averse ($\gamma \downarrow$)

Mean-Variance Optimum: Example 1

Example 1

AAPL has mean return of 10% and variance of 0.1. Risk-free saving with interest rate 2% is also available. If you have \$1 and risk aversion is $\gamma = 2$,
(1) How much AAPL you should buy?
(2) How much utility you expect to get by optimally investing?

Mean-Variance Optimum: Example 2

Example 2

In the same situation, you find your friend investing 60% of his/her \$1 budget into `AAPL`. What is your friend's risk aversion parameter?

Mean-Variance Optimum: Example 3

Example 3

In the same situation, your budget is now $\$W = \$2$. How do you change your investment?

Next Class: Portfolio

- How do you evaluate risk-return of *combinations* of assets?
- Then, what is the optimal *combination* to hold?